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European Journal of Combinatorics 27 (2006) 841–849

European Journal of Combinatorics

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Upper bounds given by equitable partitions of a primitive association scheme

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Received 26 December 2004; accepted 10 May 2005
Available online 13 June 2005

Abstract

Suppose that (X, G) is a primitive association scheme with $|G| \geq 3$ and π is an equitable partition of (X, G) with $|\pi| < |X|$. We put $\pi^* := \{C \in \pi \mid |C| > 1\}$ and $\operatorname{supp}(\pi) := \bigcup_{C \in \pi^*} C$. In this article we prove that $|G| \leq |\operatorname{supp}(\pi)| - |\pi^*| + 1$, and show a necessary and sufficient condition for the equality to hold.

1. Introduction

Not a few of attempts to generalize basic results of permutation groups have been made in terms of coherent configuration or association schemes (see [5,8,11]). In this paper we put our focus on equitable partitions of association schemes, aiming to generalize a result derived from group theory (see [4,9] for basic concepts of permutation groups).

Here is a good point to define association schemes, whose terminology follows [11]. Let X be a non-empty finite set and G a partition of $X \times X$ which does not contain the empty set. The pair (X, G) is called an *association scheme* if it satisfies the following conditions:

(i) $1_X := \{(x, x) \mid x \in X\} \in G$.

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doi:10.1016/j.ejc.2005.05.003

(ii) For each $f \in G$ $f^* := \{(x, y) \mid (y, x) \in f\} \in G$.
(iii) For all $d, e, f \in G$ and $x, y \in X$ $|xd \cap ye^*|$ is constant whenever $(x, y) \in f$, where $zg := \{w \in X \mid (z, w) \in g\}$ for $z \in X$ and a binary relation g on X.

We assume that (X, G) is an association scheme for the remainder of this section.

Now we will describe how to associate permutation groups with association schemes where the terminology on permutation groups follows [4]. Let H be a permutation group on a finite set Ω. We denote by $\mathsf{Orb}(H)$ the set of orbits of H on Ω, and by $\mathsf{Orb}_2(H)$ the set of orbits of H induced on $\Omega \times \Omega$.

We note that $(\Omega, \mathsf{Orb}_2(H))$ is an association scheme if H is transitive on Ω. In this sense an association scheme is a purely combinatorial object for generalizing the orbitals of a transitive permutation group (see [3,5,8,11] for basic concepts of association schemes and coherent configurations).

Let π be a partition of X which does not contain the empty set. We put

$$\pi^* := \{C \in \pi \mid |C| > 1\} \text{ and } \mathrm{supp}(\pi) := \bigcup_{C \in \pi^*} C.$$

According to [6] we say that π is an *equitable* partition of (X, G) if, for all $C, D \in \pi$ and $g \in G$, $|xg \cap D|$ is constant whenever $x \in C$.

We note that, if H is transitive on Ω and $K \le H$, then $\mathsf{Orb}(K)$ is an equitable partition of $(\Omega, \mathsf{Orb}_2(H))$.

We say that an association scheme (X, G) is *primitive* if there is no subset Y of X such that $1 < |Y| < |X|$ and $Yg \subseteq Y$ for some $g \in G^\times$ where $ZD := \bigcup_{z\in Z} \bigcup_{d \in D} zd$ and $D^\times := D - \{1_X\}$ for $Z \subseteq X$ and $D \subseteq G$.

We note that $(\Omega, \mathsf{Orb}_2(H))$ is a primitive association scheme if H is primitive.

We assume that H is primitive on Ω for the remainder of this section. We pick out a few of the results from among those characterizing H or bounding $|\Omega|$ by $m := |\mathrm{supp}(x)|$ where $x \in H$ and

$$\mathrm{supp}(x) := \{\alpha \in \Omega \mid \alpha^x \neq \alpha\} :$$

if $|\mathrm{supp}(x)| = 2$, then $H = \mathrm{Sym}(\Omega)$; if $|\mathrm{supp}(x)| = 3$, then $H \ge \mathrm{Alt}(\Omega)$; if $1 < m$ and H is not 2-transitive, then $|\Omega| < 4m^2$ (these results are all recorded in [4, 3.3A, 5.3A]).

In [7] the first two results can be generalized in terms of association schemes as follows:

Theorem 1.1 ([7]). *Let (X, G) be a primitive association scheme and π a equitable partition of (X, G). Suppose that there exists $C \in \pi^*$ such that $|C|$ is prime to $|D|$ for each $D \in \pi^*$ with $C \neq D$. Then $|G| = 2$.*

For the third one we can give an upper bound for $|\Omega|$ as an immediate consequence from [2, Thm. 6.14] as follows:

Theorem 1.2 ([2, Thm. 6.14]). *Let (X, G) be a primitive association scheme with $|G| \ge 3$. Then $D(x, y) > (\sqrt{|X|} - 1)/2$ for all $x, y \in X$ with $x \neq y$, where $D(x, y) := \{z \mid r(x, z) \neq r(y, z)\}$ and $r(x, y) \in G$ with $(x, y) \in r(x, y)$.*

Corollary 1.3. *Let (X, G) be a primitive association scheme with $|G| \ge 3$ and π an equitable partition of (X, G). If $m := |\mathrm{supp}(\pi)| > 1$, then $|X| < (2m + 1)^2$.*

Proof. Take $C \in \pi^*$ and $x, y \in C$ with $x \neq y$. Since $D(x, y) \leq |\mathrm{supp}(\pi)| = m$, it follows from Theorem 1.2 that $|X| \leq (2m+1)^2$. $\square$

We emphasize that the proofs of Theorems 1.1 and 1.2 are purely combinatorial.

On the other hand, it is also common to bound the rank of H by the type of a nonidentity element of H. For example, if x is a nonidentity element of H with c nontrivial disjoint cycles, then the rank of H is at most $m-c+1$ where m is the size of the support of x (see [4, 3.3.6]); it is conjectured that the rank of H is at most $1 + |\{O \in \mathrm{Orb}(K) \mid |O| > 1\}|$ for each $K \leq H_\alpha$ where $\alpha \in \Omega$ (see [10, p. 12]). In this paper we aim to generalize the first result to association schemes to approach the conjecture from a combinatorial viewpoint.

The following is our main result:

Theorem 1.4. *Let (X, G) be a primitive association scheme and π an equitable partition of (X, G) with $|\pi| < |X|$. Then $|G| \leq |\mathrm{supp}(\pi)| - |\pi^*| + 1$ with the equality holding if and only if $|X| = p$ is a prime and $G = \mathrm{Orb}_2(D_p)$ with $\pi = \{xg \mid g \in G\}$ or $\mathrm{Orb}_2(C_p)$ with $\pi = \{X\}$ where D_p is a dihedral group of degree p and C_p is a cyclic group of order p.*

We note that the above theorem is a generalization of [4, 3.3.6].

The conjecture given by Neumann can be translated to one in terms of association schemes as follows: Under the same assumption as for Theorem 1.4, $|G| \leq 1 + |\pi^*|$ whenever $\pi \neq \pi^*$. We note that the conjecture is true if $|\pi^*| = 1$ via Theorem 1.1.

2. Preliminaries

According to [7] or [11] we prepare some terminology related to association schemes. For the remainder of this section we assume that (X, G) is an association scheme.

For each $(x, y) \in X \times X$ there exists a unique element in G which contains (x, y). We shall write such a unique element as $r(x, y)$. For all subsets $Y, Z \subseteq X$ we set $r(Y, Z) := \{r(y, z) \mid y \in Y, z \in Z\}$ and $r(Y) := r(Y, Y)$. For each $g \in G$ and $Y, Z \subseteq X$ we set $g_{Y,Z} := g \cap (Y \times Z)$ and $g_Y := g_{Y,Y}$.

For each $g \in G$ we define a matrix A_g called the *adjacency matrix* of g as follows:

$$(A_g)_{x,y} := \begin{cases} 1 & \text{if } (x, y) \in g \\ 0 & \text{otherwise} \end{cases}$$

where the rows and columns of A_g are indexed by the elements of X.

Recall the definition of association schemes. We denote the constant $|xd \cap ye^*|$ with $(x, y) \in f$ by a_{def}, and we abbreviate $a_{gg^*1_X}$ as n_g.

Remark 2.1. (i) For all $d, e \in G$ we have $A_d A_e = \sum_{f \in G} a_{\mathrm{def}} A_f$.

(ii) For each partition π of X, π is an equitable partition of (X, G) if and only if $(A_g)_{U,V}$ has a constant row-sum for all $U, V \in \pi$ and $g \in G$ where $(A_g)_{U,V}$ is the submatrix of A_g restricted to $U \times V$.

Lemma 2.1 ([8]). *For all $d, e, f \in G$ we have the following:*

(i) $n_d n_e = \sum_{f \in G} a_{\mathrm{def}} n_f$;
(ii) $a_{\mathrm{def}} n_f = a_{fe^*d} n_d = a_{d^*fe} n_e$;

(iii) $\mathrm{l.c.m.}(n_d, n_e) \mid a_{def} n_f$;
(iv) $n_d = n_{d^*}$ *and* $n_{1_X} = 1$.

In [11] the *complex product* DE of two subsets $D, E \subseteq G$ is defined as follows:

$$DE := \left\{ f \in G \mid \sum_{d \in D, e \in E} a_{def} > 0 \right\}.$$

Note that the complex product is an associative binary operation on the power set of G, and we have $Y(DE) = (YD)E$ for each $Y \subseteq X$ and $D, E \subseteq G$; furthermore, $(DE)^* = E^* D^*$ where $F^* := \{ f^* \mid f \in F \}$ for $F \subseteq G$.

A nonempty subset H of G is called *closed* if $HH \subseteq H$.

Remark 2.2. (i) The singleton $\{1_X\}$ and G are closed subsets of G, called *trivial*.
(ii) (X, G) is primitive if and only if G has only trivial closed subsets.
(iii) Any closed subset H of G induces an equivalence relation R_H on X by $(x, y) \in R_H \iff x \in yH$.

The next lemma is just a translation of a result in generalized table algebras (see [1]) into our notation, and we attach a translated proof in order to make this article self-contained.

Lemma 2.2. *For each subset F of G the subset $\mathsf{L}(F) := \{g \in G \mid gF \subseteq F\}$ is closed and $n_{\mathsf{L}(F)}$ divides n_F where $n_E := \sum_{e \in E} n_e$ for $E \subseteq G$.*

Proof. Since $1_X F = F$, $\mathsf{L}(F) \neq \emptyset$. Since the complex product is associative, it is clear that $\mathsf{L}(F)$ is closed.

Let $x \in X$ and $y \in xF^*$. If $g \in \mathsf{L}(F)$, then $g^* \in \mathsf{L}(F)$ and $g^* F \subseteq F$; hence, $F^* g \subseteq F^*$. Thus, $yg \subseteq xF^* g \subseteq xF^*$. This implies that xF^* is partitioned into $\{y\mathsf{L}(F) \mid y \in xF^*\}$ by Remark 2.2(iii). Therefore, we conclude from $|y\mathsf{L}(F)| = n_{\mathsf{L}(F)}$ and $|xF^*| = n_F$ that n_F is divided by $n_{\mathsf{L}(F)}$. □

Lemma 2.3. *We have the following:*

(i) *If $a_{gg^*f} = n_g$ for some $f \in G^\times$, then $A_f A_g = n_f A_g$ and (X, G) is not primitive.*
(ii) *If $a_{gg^*f} = n_g - 1 > 0$ for some $f \in G^\times$, then $A_g A_{g^*} = n_g A_{1_X} + (n_g - 1) A_f$, $n_g = n_f$ and $A_f A_g = (n_g - 1) A_g + a_{fgh} A_h$ for some $h \in G$.*
(iii) *If $a_{gg^*g} = n_g - 1$, then $\{1_X, g\}$ is a closed subset of G.*

Proof. (i) By Lemma 2.1(ii),

$$a_{fgg} = n_f / n_g a_{gg^*f} = n_f.$$

By Lemma 2.1(i),

$$n_f n_g = \sum_{h \in G} a_{fgh} n_h \geq a_{fgg} n_g = n_f n_g.$$

This implies that $A_f A_g = n_f A_g$. By Lemma 2.2, $\mathsf{L}(\{g\})$ is a closed subset containing $f \in G^\times$ and $n_{\mathsf{L}(\{g\})}$ divides $n_g < |X|$. Thus, $\mathsf{L}(\{g\})$ is nontrivial closed subset, so the latter statement follows from Remark 2.2(ii).

(ii) By Lemma 2.1(iii), (iv), n_g divides $a_{gg^*f}n_f = (n_g - 1)n_f$. Since n_g is prime to $n_g - 1 > 0$, n_g divides n_f. If $n_f = n_g m$ for some $m \in \mathbb{N}$, then, by Lemma 2.1(i), (iv),

$$n_g n_{g^*} \geq a_{gg^*1_X} n_{1_X} + (n_g - 1)n_f = n_g + (n_g - 1)n_g m.$$

It follows from $n_g > 1$ that $n_g = n_f$ and $A_g A_{g^*} = n_g A_{1_X} + (n_g - 1)A_f$. Since $a_{fgg} = n_f/n_g a_{ggg^*f} = n_g - 1$, it follows from Lemma 2.1(i), (iii) that $A_f A_g = (n_f - 1)A_g + a_{fgh}A_h$ for some $h \in G$.

(iii) Since $A_g A_{g^*}$ is symmetric, it follows from (ii) that $g = g^*$ and $gg = \{1_X, g\}$, which is closed by definition. $\square$

Lemma 2.4 ([8, pp. 71–72]). *Let (X, G) is a primitive association scheme. If $n_g = 1, 2$ for some $g \in G^\times$, then $|X| = p$ is a prime and $G = \mathrm{Orb}_2(C_p)$, $\mathrm{Orb}_2(D_p)$, respectively, where C_p is a regular permutation group on X and D_p is a dihedral group on X.*

3. Proofs

For the remainder of this section we assume that (X, G) is a primitive association scheme with an equitable partition π of (X, G) with $|\pi| < |X|$.

If $n_g = 1$ for some $g \in G^\times$, then, by Lemma 2.4, $G = \mathrm{Orb}_2(C_p)$. From an easy observation we obtain that (X, G) has the only trivial equitable partitions, namely, the set of singletons and $\{X\}$. If $\pi = \{X\}$, then $|\mathrm{supp}(\pi)| = |X|$ and $|\pi^*| = 1$, which satisfies the inequality in Theorem 1.4.

If $|G| = 2$, then the inequality is also satisfied since $|\mathrm{supp}(\pi)| > |\pi^*|$.

Thus, we may assume that $n_g \geq 2$ for each $g \in G^\times$, and $|G| \geq 3$.

For short we set $S := \mathrm{supp}(\pi)$.

Lemma 3.1. *If $x, y \in X$ lie in the same cell, then $xg - S = yg - S$ for each $g \in G$.*

Proof. If $z \in X - S$, then $\{z\} \in \pi - \pi^*$, so $|xg \cap \{z\}| = |yg \cap \{z\}|$. This implies that $xg - S = yg - S$. $\square$

The following is an immediate consequence of Lemma 3.1:

For each $g \in G$ and all $x, y \in C$ with $r(x, y) = f$ and $C \in \pi^*$,

$$a_{gg^*f} = |xg \cap yg \cap S| + |xg - S|. \tag{1}$$

In particular, if $x = y$, then

$$n_g = |xg \cap S| + |xg - S|. \tag{2}$$

Lemma 3.2. *If $x \in S$, then $G = \{r(x, s) \mid s \in S\}$.*

Proof. Let $C \in \pi^*$ with $x \in C$. Since π is an equitable partition, it suffices to show that $G = r(C, S)$. Suppose not, i.e., there exists $g \in G - r(C, S)$. Since $g_{C,S} = \emptyset$, it follows from Lemma 3.1 that $xg = yg$ for all $y \in C$. Taking $x, y \in C$ with $x \neq y$ and $f := r(x, y)$ we obtain that

$$a_{gg^*f} = |xg \cap yg| = |xg| = n_g.$$

By Lemma 2.3(i), (X, G) is not primitive, a contradiction. $\square$

For the remainder of this section we fix $C \in \pi^*$, $x, y \in C$ with $x \neq y$, and $f := r(x, y)$.

We define a simple graph whose vertex set is S, and whose edge set I is given by the following. Two vertices $u, v \in S$ are adjacent if and only if $r(x, u) = r(x, v)$ and $u \neq v$. Clearly, the graph (S, I) is a disjoint union of cliques. Since π^* is a partition of S, it induces a disjoint union of cliques, say (S, K); i.e., $u, v \in S$ are adjacent in (S, K) if and only if $u, v \in D$ for some $D \in \pi^*$ and $u \neq v$.

Remark 3.1. We remark that $u \in S$ has degree j in (S, I) if and only if $|xr(x, u) \cap S| = j + 1$.

Lemma 3.3. *Suppose that $u \in S$ has degree zero in (S, I). Then $r(C)^\times = \{f\}$ and $a_{r(x,u)r(x,u)^* f} = n_f - 1$.*

Proof. For short we set $g := r(x, u)$. By Remark 3.1, $|xg \cap S| = 1$. Then, by (1) and (2),

$$a_{gg^* f} = |xg \cap yg \cap S| + |xg - S| \geq n_g - 1.$$

By Lemma 2.3(i) and $a_{gg^* f} \leq n_g$, $a_{gg^* f} = n_g - 1$. By Lemma 2.3(ii), $gg^* = \{1_X, f\}$ and $n_g = n_f$. Since $n_g > 1$, it follows from Lemma 3.1 that there exists $z \in ug - S = vg - S$ for all $u, v \in C$. This implies that $r(u, v) \in gg^* = \{1_X, f\}$. □

Lemma 3.4. *For all u, v in a cell of π^* we have $r(x, u) \in r(C)r(x, v)$.*

Proof. Let $g := r(x, u)$ and $e := r(x, v)$. Since $|ue^* \cap C| = |ve^* \cap C| \geq 1$, there exists $w \in C$ such that $w \in ue^*$. Then $w \in xr(C) \cap ue^*$. This implies that $g = r(x, u) \in r(C)e = r(C)r(x, v)$. □

Lemma 3.5. *For each $D \in \pi^*$, there are no two distinct vertices in D which have degree zero in (S, I).*

Proof. Suppose that $u, v \in D$ have degree zero in (S, I) with $u \neq v$. Without loss of generality we may assume that $g := r(x, u) \neq 1_X$. Then, by Lemma 3.3, $r(C)^\times = \{f\}$ and $a_{gg^* f} = n_f - 1$.

If $D = C$, then $f = g$ and $a_{gg^* g} = n_g - 1$. By Lemma 2.3(iii), $G = \{1_X, g\}$, a contradiction to the assumption.

If $D \neq C$, then $e := r(x, v) \neq 1_X$ and $a_{ee^* f} = n_f - 1$. Applying Lemma 2.3(ii) for g and e we obtain that $fg = \{g, h\}$ for some $h \in G$ and $fe = \{e, h'\}$ for some $h' \in G$. By Lemma 3.4, $g \in r(C)e = \{e, h'\}$ and $e \in r(C)g = \{g, h\}$. Note that $e \neq g$ since u and v have degree zero. Thus, $g = h'$, $e = h$, and $f\{g, e\} \subseteq \{g, e\}$. Since (X, G) is primitive, it follows from Lemma 2.2 that $\{g, e\} = G$, which contradicts $1_X \notin \{g, e\}$. □

Let T be a connected component of the graph $(S, I \cup K)$ consisting of exactly $l(T)$ cells in π^*. We denote by (T, I_T) the subgraph of (S, I) induced by T. Then we have

$$|I_T| \geq l(T) - 1. \tag{3}$$

Recall that a connected graph with v vertices and $v - 1$ edges forms a tree. The following remark is obtained from observation on (T, I_T):

Remark 3.2. If the equality holds in (3), then we have the following:

(i) Each vertex in T has degree at most one in (T, I_T).
(ii) There exists at most one edge between two distinct cells and no edge on one cell in (T, I_T).
(iii) There is no sequence $(D_1, D_2, \dots, D_r)$ of cells in T such that each D_i is connected to both D_{i-1} and D_{i+1} by one edge in I_T where the subscripts are read modulo r.

Proposition 3.6. *If the equality holds in* (3)*, then* $l(T) = |\pi^*|$ *and* $n_f = 2$.

Proof. By Remark 3.2, there exist at least two pairs of $(u, D) \in T \times \pi^*$ such that $u \in D$ and u is a unique vertex of degree one in D. So, we can take such a pair (u, D) with $C \neq D$ and $u \neq x$. By Remark 3.2 and Lemma 3.5, $|D| = 2$. We set $x_1 \in D - \{u\}$ and $x_2 := u$. By Lemma 3.3, $r(C)^\times = \{f\}$.

Suppose that, for each $2j$ with $1 \le 2j \le 2i$, $\{x_{2j-1}, x_{2j}\} \in \pi^*$ and x_{2j-1} is a unique neighbour of x_{2j-2} in (T, I_T). Then we claim that, if x_{2i+1} is a unique neighbour of x_{2i} in (T, I_T), then the cell containing x_{2i+1} has size two. We set $g_0 := r(x, x_{2i-1})$ and $g_1 := r(x, x_{2i})$, so that $g_1 = r(x, x_{2i+1})$. By Remark 3.1, $|xg_1 \cap S| = 2$. By Lemma 3.1, $a_{g_1 g_1^* f} \ge n_{g_1} - 2$. By Lemma 2.1(i), (ii),

$$n_f n_{g_1} = (a_{g_1 g_1^* f} n_f / n_{g_1}) n_{g_1} + a_{f g_1 g_0} n_{g_0} + \sum_{d \in G - \{g_0, g_1\}} a_{f g_1 d} n_d. \tag{4}$$

Note that $a_{f g_1 g_0} > 0$ by Lemma 3.4 with $r(C)^\times = \{f\}$. Since $a_{g_1 g_1^* f} \ge n_f - 2$, it follows from (4) and Lemma 2.1(iii) that $\sum_{d \in G - \{g_1, g_2\}} a_{f g_1 d} n_d \le n_f$. By Lemma 2.1(iii), $n_f \mid a_{f g_1 d} n_d$. Thus, there exists at most one $d \in G - \{g_0, g_1\}$ such that $a_{f g_1 d} > 0$. Let $E \in \pi^*$ with $x_{2i+1} \in E$ and $v \in E - \{x_{2i+1}\}$. Then $r(x, v) \notin \{g_1, g_0\}$ by Remark 3.2(ii). Thus, there exists $g_2 \in G - \{g_0, g_1\}$ such that $a_{f g_1 g_2} > 0$. Applying Lemma 3.4 for E we obtain from $r(x, v) \notin \{g_1, g_0\}$ that $r(x, v) \in r(C) g_1 = \{g_0, g_1, g_2\}$ and $r(x, v) = g_2$. The claim follows from Remark 3.2(ii).

Applying the above claim we can take a sequence $\{x_i\}_{i=1}^{2l}$ with $l := l(T)$ such that x_{2l} and x_1 have degree zero in (T, I_T) and $\{x_{2j-1}, x_{2j}\} \in \pi^*$ for each j with $1 \le j \le l$. Since x_{2l} has degree zero in (T, I_T), it follows from Lemma 3.3 that $n_{g g^* f} = n_f - 1$ where $g := r(x, x_{2l})$. By Lemmas 2.3(ii) and 3.4, we have $fg \subseteq \{g, r(x, x_{2l-1})\}$. It follows that

$$f\{r(x, x_i) \mid 1 \le i \le 2l\} \subseteq \{r(x, x_i) \mid 1 \le i \le 2l\}.$$

By Lemma 2.2, $G = \{r(x, x_i) \mid 1 \le i \le 2l\}$. If T is a proper subset of S, then, for $w \in S - T$, there exists x_i such that $r(x, x_i) = r(x, w)$, which contradicts that T is a connected component in $(S, I \cup K)$. Thus, $l(T) = \pi^*$.

Note that $x_{2l} = x$ since x has degree zero and $\{x_1, x_{2l}\}$ are unique vertices of degree zero. Thus, $y = x_{2l-1}$ and $f = r(x, x_{2l-1}) = r(x, x_{2l-2})$. Since $xf \cap S = \{y, x_{2l-2}\}$ and $yf \cap S = \{x, x_{2l-3}\}$, we have $|xf \cap yf \cap S| = 0$. It follows from (2) and (1) that $a_{ff^*f} = |xf - S| = n_f - 2$. By Lemma 2.1(i), (iii),

$$ff^* = \{1_X, f, d\} \text{ for some } d \in G \text{ with } a_{ff^*d} n_d = n_f. \tag{5}$$

Since $r(y, x_{2l-2}) \in ff^* - \{1_X, f\}$ by the above claim, $r(y, x_{2l-2}) = d$.

On the other hand,

$$a_{ff^*f} = |xf \cap x_{2l-2} f| = n_f - 2.$$

Note that $xf \cap S = \{y, x_{2l-2}\}$ and $x_{2l-2}f \cap S = \{x, w\}$ for some $w \in S$ with $w \neq x$. Since

$$x_{2l-1}f \cap x_{2l-2}f = (yf - S) \cap (x_{2l-2}f - S) = (xf - S) \cap (x_{2l-2}f - S),$$

we have $|xf - S| = n_f - 2$, $x_{2l-2}f - S = xf - S = yf - S$. Thus, $a_{ff^*d} = |yf \cap x_{2l-2}f| = n_f - 1$. It follows from (5) that $(n_f - 1)n_d = n_f$. Since n_f is prime to $n_f - 1$, $n_f \mid n_d$, and, hence, $n_f = 2$. □

Proof of Theorem 1.4.

Proof. It follows from Lemma 3.2 that $|G|$ is the number of connected components in (S, I). If $|I_T| = l(T) - 1$ for a connected component T in $(S, I \cup K)$, then, by Proposition 3.6, $l(T) = |\pi^*|$ and $S = T$. Since I_T is a disjoint union of edges, $|G| = |S| - |I| = |S| - (|\pi^*| - 1)$.

Suppose that $|I_T| \geq l_T$ for each connected component in $(S, I \cup K)$. Then $|G| \leq |S| - \sum_T l(T) = |S| - |\pi^*|$ where T ranges over all the connected components of $(S, I \cup K)$.

Assume that the equality holds. Then, by the previous paragraphs and Proposition 3.6, $n_f = 2$. It follows from Lemma 2.4 that $|X| = p$ is a prime $G = \mathrm{Orb}_2(D_{2p})$. In the proof of Proposition 3.6, we see that π^* consists of cells of size two. Thus, $\frac{|X|-1}{2} + 1 = |G| = 2|\pi^*| - |\pi^*| + 1$. It follows that $|X| = 2|\pi^*| + 1$, and, hence, π consists of π^* and a singleton $\{z\}$. Note that, for $w \in X$ with $\{w, w'\} \in \pi^*$, we have $|wr(w, z) \cap \{z\}| = |w'r(w, z) \cap \{z\}| = 1$. This implies that $\pi = \{zg \mid g \in G\}$.

The converse is obtained from easy observation. □

Acknowledgements

The research activity of the author is supported by Korea Science and Engineering Foundation (R05-2003-000-12084-0). The author would like to express his gratitude for the support.

The author would like to thank Mr. Kijung Kim for his careful reading and checking of errors in the proofs.

In the editing process the article was improved very much. The following are due to comments from an anonymous referee:

(i) Inserting [2,5,8,10] in the references.
(ii) Connection to a conjecture given by P. Neumann.
(iii) In the original article the author showed the inequality of $|X| < (m-1)^{2m}$ under the same assumption as for Corollary 1.3, spending a few pages. A referee informed the author of [2], showing a way to improve the upper bound.

As is seen above, the referee reports were so valuable that the author was informed of a huge amount of background lying behind the topic that the author suggested in this paper. The author would like to express his deepest gratitude to the anonymous referees.

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